Stability conditions on quiver representations

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1 Introduction

These notes are written for the Bridgeland stability learning seminar at Harvard. The goal is to discuss stability conditions on quiver representations. These notes are heavily based on [KJ16] (for basic background on quivers) and [Bal].

1.1 Conventions

We always work over the ground field $\mathbb{C}$.

2 Quiver representations

2.1 Quivers

First, a brief overview of generalities on quivers. For more details, see [KJ16].

Definition 2.1 (quiver): A **quiver** is a directed graph.

We typically denote a quiver by $\overrightarrow{Q}$. We'll denote the set of vertices by I and the set of directed edges by Ω. For an edge $e \in \Omega$, we'll write $e : i \to j$ to indicate that it goes from i to j, and write e_h to denote the head of e, and e_t to denote the tail of e. If we forget the directions of the edges in $\overrightarrow{Q}$, then we get a graph, which we will denote by Q.

For our purposes, we always assume that I, Ω are **finite**, and that $\overrightarrow{Q}$ is **connected**.

Definition 2.2 (acyclic): We say a quiver $\overrightarrow{Q}$ is **acyclic** if it has no oriented cycles.

2.2 Representations of quivers

Definition 2.3 (representation): Let $\overrightarrow{Q}$ be a quiver. Then a **representation** V of $\overrightarrow{Q}$ is the data of:

- Vector spaces V_i for each $i \in I$,
- Linear maps $x_e : V_i \to V_j$ for each $e : i \to j \in \Omega$.

We only consider **finite-dimensional representations** in these notes.

Definition 2.4: A **morphism of representations** $f : V \to W$ is a collection of operators $f_i : V_i \to W_i$ for each $i \in I$, which commute with the operators x_e. The morphisms $V \to W$ form a vector space and we denote it by $\mathrm{Hom}_{\overrightarrow{Q}}(V, W)$, or simply by $\mathrm{Hom}(V, W)$.

Definition 2.5 (Rep $\overrightarrow{Q}$): These two combine to give us the **category of (finite-dimensional) representations** of a quiver $\overrightarrow{Q}$, which we denote by $\mathrm{Rep}\,\overrightarrow{Q}$. We denote its bounded derived category by $D^b(\overrightarrow{Q}) \coloneqq D^b(\mathrm{Rep}\,\overrightarrow{Q})$.

$\mathrm{Rep}\,\overrightarrow{Q}$ has many of the standard operations that we're familiar with; these include direct sums, subrepresentations, quotient representations, kernels, and images. This makes $\mathrm{Rep}\,\overrightarrow{Q}$ into an $\mathbb{C}$-**linear abelian category**. Actually, the reason for this is due to the path algebra.

2.3 Path algebra

Fix a quiver $\overrightarrow{Q}$. A **path** in $\overrightarrow{Q}$ is just a sequence of edges so that the tail of one edge is the head of the next. The **length** of a path is just the number of edges in the sequence. We allow for length 0 paths, which we denote by e_i for $i \in I$. Finally, we define **multiplication of paths** by concatenating them if the tail of first path is the head of the second path, and zero otherwise.

Definition 2.6 (path algebra): The **path algebra** $\mathbb{C}\overrightarrow{Q}$ is the $\mathbb{C}$-algebra with basis given by all paths in $\overrightarrow{Q}$ (including length 0), with multiplication given by multiplication of paths.

Here are some immediate properties:

a) $\mathbb{C}\overrightarrow{Q}$ is an associative algebra with unit $1 = \sum_{i \in I} e_i$.

b) $\mathbb{C}\overrightarrow{Q}$ is naturally $\mathbb{Z}_{\geq 0}$-graded by path length.

c) $\mathbb{C}\overrightarrow{Q}$ is finite-dimensional iff $\overrightarrow{Q}$ contains no oriented cycles.

d) The length-zero paths e_i are indecomposable projections summing to 1.

The important point is that:

Theorem 2.7: The category of $\overrightarrow{Q}$-representations, not necessarily finite-dimensional, is equivalent to the category of left $\mathbb{C}\overrightarrow{Q}$-modules.

If we start with a $\overrightarrow{Q}$-representation V, then we get a $\mathbb{C}\overrightarrow{Q}$-module M by setting $M \coloneqq \bigoplus_{i \in I} V_i$, with the path $e \in \Omega$ acting by x_e. On the other hand, from a $\mathbb{C}\overrightarrow{Q}$-module M, we recover a $\overrightarrow{Q}$-representation V by setting $V_i \coloneqq e_i M$, and for $e : i \to j \in \Omega$, setting x_e to be the operator induced by the path e, sending $e_i M \to e_j M$ since $e = e_j \cdot e \cdot e_i \in \mathbb{C}\overrightarrow{Q}$.

Therefore, we easily see that Rep $\overrightarrow{Q}$ is abelian and inherits all the usual notions from the theory of modules over associative algebras. In particular:

Definition 2.8: We have notions of **simple**, **semisimple**, and **indecomposable** representations in Rep $\overrightarrow{Q}$. Write $\operatorname{Ind}(\overrightarrow{Q})$ to be the set of isomorphism classes of nonzero indecomposable representations of Rep $\overrightarrow{Q}$.

Theorem 2.9: The simple representations of an acyclic quiver are $S(i)$ for $i \in I$, which are the representations given by a single one-dimensional vector space at vertex i, zero for every other vertex, and every edge is the zero operator.

Proof. Suppose V is a simple representation. Pick some vertex $i \in I$ such that $V_i \neq 0$, and i is "maximal" in the sense that for every edge $i \longrightarrow j$, then $V_j = 0$. This can be done because there are no oriented cycles. Then V_i itself is a subrepresentation (change every other vector space to 0 and every edge to the zero operator; we also are abusing notation here). By simplicity of V, it must be true that $V = V_i$.
On the other hand, it's obvious that every $S(i)$ is simple. □

So the simple representations are easy to describe, which is good because of theorems like Jordan-Hölder. However, the indecomposable representations are also very important: every finite-dimensional representation decomposes as a direct sum of indecomposables, uniquely up to reordering. We can describe the indecomposable projectives fairly easily as well; the full list of indecomposables is more complicated, see [KJ16].

Definition 2.10 ($P(i)$): Define the representations $P(i)$ for $i \in I$ to be the $\overrightarrow{Q}$-representation associated to the left $\mathbb{C}\overrightarrow{Q}$-module $(\mathbb{C}\overrightarrow{Q})e_i$, spanned by all paths starting at i.

Note that the $P(i)$ are clearly projective, as $\mathbb{C}\overrightarrow{Q} = \bigoplus_{i \in I} P(i)$. They're characterized by the fact that for any $\overrightarrow{Q}$-representation V, we have $\operatorname{Hom}_{\overrightarrow{Q}}(P(i), V) = V_i$.

Theorem 2.11: Assume $\overrightarrow{Q}$ is acyclic. Then $\{P(i) \mid i \in I\}$ are the full list of nonzero projective indecomposables in Rep $\overrightarrow{Q}$.

2.4 Grothendieck group

Definition 2.12 ($K(\overrightarrow{Q})$): Let $K(\overrightarrow{Q}) := K(\operatorname{Rep} \overrightarrow{Q})$, the Grothendieck group of the abelian category Rep $\overrightarrow{Q}$.

Definition 2.13 (graded dimension): Define the **graded dimension** $\mathbf{dim}\, V \in \mathbb{Z}^I$ to be the $|I|$-tuple given by $(\mathbf{dim}\, V)_i = \dim V_i$.

Theorem 2.14: Let $\overrightarrow{Q}$ be acyclic. Then the graded dimension map $\mathbf{dim}$ induces an isomorphism $K(\overrightarrow{Q}) \xrightarrow{\sim} \mathbb{Z}^I$.

Definition 2.15: Define the number $\langle V, W \rangle$ of two $\overrightarrow{Q}$-representations V, W to be

$$\langle V, W \rangle := \sum_i (-1)^i \dim \operatorname{Ext}^i(V, W) = \chi(\mathbf{R}V, W)).$$

Remark 2.16: It's known that we can always take a two-step projective resolution of any $\overrightarrow{Q}$-representation, hence the category Rep $\overrightarrow{Q}$ is **hereditary**, i.e. all $\operatorname{Ext}^{>1}$ vanish.

It turns out that the Euler form is *very* insensitive to the representation itself.

Theorem 2.17: The number $\langle V, W\rangle$ depends only on the graded dimensions of the representations, and hence descends to a bilinear form on $\mathbb{Z}^I$, called the **Euler form**. In fact, for $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^I$,

$$\langle \mathbf{v}, \mathbf{w}\rangle = \sum_{i\in I} \mathbf{v}_i \cdot \mathbf{w}_i - \sum_{e: i\to j\in\Omega} \mathbf{v}_i \mathbf{w}_j.$$

Note that the Euler form is not symmetric, so we'll frequently use the **symmetrized Euler form**

$$(\mathbf{v}, \mathbf{w}) \coloneqq \langle \mathbf{v}, \mathbf{w}\rangle + \langle \mathbf{w}, \mathbf{v}\rangle.$$

Remark 2.18: Note that the symmetrized Euler form is independent of the orientation of $\overrightarrow{Q}$.

3 Stability conditions

Fix a finite acyclic quiver $\overrightarrow{Q}$. We want to study stability conditions on $\operatorname{Rep} \overrightarrow{Q}$.

3.1 Moduli space of $\overrightarrow{Q}$-representations

In order to discuss stability conditions on $\overrightarrow{Q}$-representations, we need to enumerate all isomorphism classes of them. We know that the class of a $\overrightarrow{Q}$-representation V depends only on its graded dimension $\mathbf{dim}\, V \in \mathbb{Z}^I$; however, there may be many isomorphism classes of such representations. So let's fix some $\mathbf{v} \in \mathbb{Z}^I$ and study all $\overrightarrow{Q}$-representations with graded dimension $\mathbf{v}$.

Let's consider what such a V would look like. We know that $\dim V_i = \mathbf{v}_i$, so $V_i = \mathbb{C}^{\mathbf{v}_i}$. It only remains to parametrize the morphisms between the V_i. So define

$$\mathcal{R}_{\mathbf{v}} \coloneqq \bigoplus_{e: i\to j\in\Omega} \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}^{\mathbf{v}_i}, \mathbb{C}^{\mathbf{v}_j}).$$

However, each isomorphism class of representation appears many times; isomorphisms are given by invertible maps $V_i \xrightarrow{\sim} V_i$ for all $i \in I$, so we need to quotient by this. Define

$$\mathrm{GL}_{\mathbf{v}} \coloneqq \prod_{i\in I} \mathrm{GL}_{\mathbf{v}_i}.$$

Then $\mathrm{GL}_{\mathbf{v}}$ naturally acts on $\mathcal{R}_{\mathbf{v}}$ by conjugation:

$$(g_i)_{i\in I} \cdot (\varphi_e)_{e: i\to j\in\Omega} = (g_j \varphi_e g_i^{-1})_{e: i\to j}.$$

It's clear that

$$\{\mathrm{GL}_{\mathbf{v}}\text{-orbits in } \mathcal{R}_{\mathbf{v}}\} \longleftrightarrow \{\text{isomorphism classes of } \overrightarrow{Q}\text{-representations with graded dimension } \mathbf{v}\}.$$

So we need to make sense $\mathcal{R}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}$, or whatever is the appropriate analogue of that quotient in the world of varieties.

3.2 GIT quotients

Let G be a reductive algebraic group acting algebraically on an affine algebraic variety M. Of course, in our setup, we take $\mathrm{G} = \mathrm{GL}_{\mathbf{v}}$ and $M = \mathcal{R}_{\mathbf{v}}$. This subsection explains GIT quotients; we won't *really* need it for studying stability conditions on quiver representations, since the main focus is actually on *twisted* GIT quotients (see §3.3), but twisted GIT quotients are in some sense a generalization of GIT quotients, so this subsection may be helpful to the reader.

The idea is that we want to build a moduli space for the G-orbits in M, i.e., a scheme version of M/G (which is a perfectly reasonable topological space, but rarely has many useful properties beyond that). This is actually rather hard, because the orbits come in many varying sizes and shapes. If we want the moduli space to actually be a scheme (even an affine variety) so that we can do our best with constructing quotients in the category of schemes (or affine varieties), we'll need to compromise and give up a lot.

Definition 3.1 (GIT quotient): We define the **GIT quotient** $M /\!\!/ \mathrm{G} := \operatorname{Spec} \mathbb{C}[M]^{\mathrm{G}}$.

This is supposed to be our scheme version of the topological quotient M/G. It is indeed a scheme, and even an affine variety (the algebra $\mathbb{C}[M]^{\mathrm{G}}$ is finitely-generated due to HIlbert). However, topologically, the points of $M /\!\!/ \mathrm{G}$ are only the **closed orbits** in M, not all orbits. There's a natural topological map $M/\mathrm{G} \to M /\!\!/ \mathrm{G}$ (sending a G-orbit $\mathbb{O}$ to the maximal ideal in $\mathbb{C}[M]^{\mathrm{G}}$ of functions vanishing on $\mathbb{O}$). However, whenever two orbits $\mathbb{O}, \mathbb{O}' \in M/\mathrm{G}$ "intersect," i.e., $\overline{\mathbb{O}} \cap \overline{\mathbb{O}'} \neq \emptyset$, then they get identified in $M /\!\!/ \mathrm{G}$. One way to understand this is that the G-orbits form a stratification of M, hence there's a partial order on the orbits where $\mathbb{O} \leq \mathbb{O}' \longleftrightarrow \overline{\mathbb{O}} \subset \overline{\mathbb{O}'}$, and the closed orbits are exactly the minimal elements of this partial ordering. The GIT quotient thus only remembers the minimal elements, i.e., the closed sets.

Example 3.2: Let $\mathbb{G}_m \curvearrowright \mathbb{A}^2$ by the standard scaling action on both coordinates. We have many orbits: namely, we have the unique closed orbit $\{(0,0)\}$, and then we have a ton of dimension one orbits indexed by the ratio $(a,b) \mapsto b/a$. However, the GIT quotient only cares about the closed orbits; here, there's only one, so the GIT quotient is $\operatorname{Spec} \mathbb{C}$, which is just a point. This can also be computed by checking the $\mathbb{G}_m$-invariants in $\mathbb{C}[x,y]$, for which we quickly find that there are none except the constants.
So the GIT quotient can lose quite a lot of information.

3.3 Twisted GIT quotient

Once again, let G be a reductive algebraic group acting algebraically on an affine algebraic variety M; in our setup, we take $\mathrm{G} = \mathrm{GL}_{\mathbf{v}}$ and $M = \mathcal{R}_{\mathbf{v}}$.

We will review the theory of twisted GIT quotients, which will actually be the relevant theory in our case. Let $\chi : \mathrm{G} \to \mathbb{G}_m$ be a character. Define

$$\mathbb{C}[M]^{\mathrm{G},\chi} := \{f \in \mathbb{C}[M] \mid f(g \cdot m) = \chi(g) \cdot f(m)\},$$

the **relative invariants**. We get a graded algebra

$$\bigoplus_{n \geq 0} \mathbb{C}[M]^{\mathrm{G},\chi^n},$$

and Hilbert's theorem implies that it is finitely generated.

Definition 3.3 (twisted GIT quotient): The **twisted GIT quotient** is defined to be

$$M /\!\!/_{\chi} \mathrm{G} := \operatorname{Proj}\left(\bigoplus_{n \geq 0} \mathbb{C}[M]^{\mathrm{G},\chi^n}\right).$$

The 0th graded component, $\mathbb{C}[M]^{\mathrm{G},\chi^0} = \mathbb{C}[M]^{\mathrm{G}}$ recovers the standard GIT quotient $M /\!\!/ \mathrm{G}$. Thus we get a projective morphism

$$\pi : M /\!\!/_{\chi} \mathrm{G} \to M /\!\!/ \mathrm{G}. \tag{1}$$

3.4 GIT stability

We continue the setup as in §3.3.

Definition 3.4 (GIT (semi)stability): Extend the action of G on M to an action on $M \times \mathbb{A}^1$ by $g(m, z) := (g(m), \chi^{-1}(g)z)$.
A point $x \in M$ is χ**-semistable** if for any nonzero $z \in \mathbb{C} - \{0\}$, the closure of the G-orbit of (x, z) is disjoint from the zero section $M \times \{0\}$. We denote the set of χ-semistable points of M by M^{ss}_χ.
A point $x \in M$ is χ**-stable** if it is χ-semistable, has finite stabilizer $\mathrm{G}_x \subset \mathrm{G}$, and for any nonzero z the G-orbit of (x, z) is closed in $M \times \mathbb{A}^1$. In fact, this is equivalent to the G-orbit of x being closed in M^{ss}_χ. We denote the set of χ-stable points of M by M^s_χ.

To describe these conditions more explicitly, we'll make frequent use of:

Theorem 3.5 (Geometric reductivity principle): If $X, Y \subset M$ are closed G-invariant subvarieties and $X \cap Y = \emptyset$, then there exists a G-invariant polynomial f such that $f|_X = 0$ and $f|_Y = 1$.

We immediately deduce a technical condition about χ-semistability.

Corollary 3.6: A point $x \in M$ is χ-semistable iff there exists $f \in \mathbb{C}[M]^{\mathrm{G},\chi^n}$ for some $n \geq 1$, for which $f(x) \neq 0$.

Proof. Suppose x is χ-semistable. We apply the geometric reductivity principle (3.4) to the G-action on $M \times \mathbb{A}^1$, which tells us there's a function $\widehat{f} \in \mathbb{C}[M \times \mathbb{A}^1]^{\mathrm{G}}$ such that $\widehat{f}|_{M\times\{0\}} = 0$ and $\widehat{f}|_{\overline{\mathrm{G}\cdot(x,1)}} \neq 0$. Since $\widehat{f} \in \mathbb{C}[M \times \mathbb{A}^1]^{\mathrm{G}}$, and G acts on the $\mathbb{A}^1$-component by χ^{-1}, we know that G must act correspondingly by χ on the M-coordinate; thus we can write

$$\widetilde{f}(x, z) = \sum_{n\geq 0} f_n(x) z^n, \quad f_n \in \mathbb{C}[M]^{\mathrm{G},\chi^n}.$$

Now by hypothesis $\widehat{f}|_{M\times\{0\}} = 0$, so $\widehat{f}(m, 0) = f_0(m) = 0 \implies f_0 = 0$. But since $\widehat{f}$ is not identically zero (it is nonzero on the closure of the G-orbit of $(x, 1)$), then there must be some $0 \neq f_n \in \mathbb{C}[M]^{\mathrm{G},\chi^n}$ with $f_n(x) \neq 0$.
In the other direction, if $f \in \mathbb{C}[M]^{\mathrm{G},\chi^n}$ is such that $f(x) \neq 0$, then the function $\widetilde{f} := (x, z) \mapsto f(x)\cdot z^n$ is G-invariant on $M \times \mathbb{A}^1$. Since $f(x) \neq 0$, it's clear that for any $z \neq 0$, then $\widetilde{f}(x, z) \neq 0$, hence is a nonzero constant on the entire G-orbit $\mathrm{G}\cdot(x, z)$, hence is a nonzero constant on the closure $\overline{\mathrm{G}\cdot(x, z)}$ as well. But $\widetilde{f}(m, 0) = f(m)\cdot 0^n = 0$, so $\widetilde{f}|_{M\times\{0\}} = 0$. It follows that $\overline{\mathrm{G}\cdot(x, z)} \cap M \times \{0\} = \emptyset$, so $x \in M^{ss}_\chi$. □

Corollary 3.7:

a) $M^{ss}_\chi \subset M$ is open and G-invariant (but possibly empty).

b) For $N \in \mathbb{Z}_{>0}$, $x \in M$ is χ-semistable iff it is χ^N-semistable. Thus, the notion of χ-semistable can be defined for any rational character $\chi \in X(\mathrm{G}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

c) Every $x \in M^{ss}_\chi$ defines a maximal ideal $J_x := \{f \mid f(x) = 0\} \subset \bigoplus_{n\geq 0} \mathbb{C}[M]^{\mathrm{G},\chi^n}$, and is not the irrelevant ideal. Thus we have a natural map

$$M^{ss}_\chi/\mathrm{G} \to M /\!\!/_\chi \mathrm{G}, \quad x \mapsto J_x.$$

In lieu of this, it would be nice to understand the map $M^{ss}_\chi/\mathrm{G} \to M /\!\!/_\chi \mathrm{G}$.

Theorem 3.8:

a) The map $M^{ss}_\chi/\mathrm{G} \to M /\!\!/_\chi \mathrm{G}$ is surjective.

b) Two points $x, y \in M^{ss}_\chi/\mathrm{G}$ (corresponding to semistable G-orbits $\mathbb{O}_x, \mathbb{O}_y \subset M^{ss}_\chi$) are mapped to the same point in $M /\!\!/_\chi \mathrm{G}$ iff the closures of their orbits (taken in M^{ss}_χ) intersect, i.e., $\overline{\mathbb{O}_x} \cap \overline{\mathbb{O}_y} \cap M^{ss}_\chi \neq \emptyset$.

c) As a topological space, $M /\!\!/_\chi \mathrm{G} = M^{ss}_\chi/\sim$, where $x \sim y$ iff the closures of their orbits in M^{ss}_χ intersect.

d) In fact, $M /\!\!/_\chi \mathrm{G} = \{\text{closed orbits in } M^{ss}_\chi\}$. (Note that this is weaker than being closed in M.)

Using this explicit description of $M /\!\!/_\chi \mathrm{G}$, we can explicitly describe the map $\pi : M /\!\!/_\chi \mathrm{G} \to M /\!\!/ \mathrm{G}$ from (1).

Theorem 3.9: Let $x \in M_\chi^{ss}$ and denote $[x]$ its image in $M /\!\!/_\chi \mathrm{G}$. Then

$$\pi([x]) = \text{the unique closed orbit in } M \text{ contained in } \overline{\mathbb{O}_x}.$$

Proof. Let $\mathbb{O}_1$ be the unique closed orbit in $\overline{\mathbb{O}_x}$. For $f \in \mathbb{C}[M]^{\mathrm{G}}$, we can verify that $f(\mathbb{O}_1) = \pi^* f(\mathbb{O}_x)$. □

So more or less, what's happening is that when you take a GIT quotient, you form a partial ordering on the orbits (by containment of the closure of the orbits); the closed orbits are the minimal ones, and the GIT quotient only remembers the minimal ones. So the GIT quotient $M /\!\!/ \mathrm{G}$ remembers only the smallest G-orbits in M. But the twisted GIT quotient $M /\!\!/_\chi \mathrm{G}$ only requires that the orbits are closed in $M_\chi^{ss} \subset M$; this is weaker than being closed in M, and the map $\pi : M /\!\!/_\chi \mathrm{G} \to M /\!\!/ \mathrm{G}$ "remembers" the rest of the orbit as we add back the complement $M \setminus M_\chi^{ss}$, and then sends the closed-in-M_χ^{ss}-but-not-in-M orbits to the true minimal closed orbit contained in its closure.

We'd like to say things about stable points as well. Recall that the property of being stable implies that their G-orbits intersect iff their closures intersect in M_χ^{ss}, hence by Theorem 3.4c, distinct stable orbits define distinct points in $M /\!\!/_\chi \mathrm{G}$¡ so

$$M_\chi^s \subset M /\!\!/_\chi \mathrm{G}.$$

Theorem 3.10: Assume $M_\chi^s \neq \emptyset$.

a) M_χ^s is open in M_χ^{ss}, and thus in M.

b) If M is irreducible (which it will be for us - it'll be $\mathcal{R}_{\mathbf{v}}$), then M_χ^s is dense in M_χ^{ss}, and M_χ^s/G is dense in $M /\!\!/_\chi \mathrm{G}$.

c) If M is nonsingular (again, it will be for us) and for every $x \in M_\chi^s$, the stabilizer G_x is trivial, then M_χ^s/G is a nonsingular variety of dimension $\dim M - \dim \mathrm{G}$.

Finally, we'll also make note of a numerical criterion which detects (semi)stability.

Theorem 3.11 (Mumford): A point $x \in M$ is semistable (respectively stable), iff for any one-parameter subgroup $\lambda : \mathbb{G}_m \to \mathrm{G}$ such that $\lim_{t\to 0} \lambda(t) \cdot x$ exists, then $\langle \chi, \lambda \rangle \geq 0$ (respectively, $\langle \chi, \lambda \rangle > 0$, for nontrivial λ).

3.5 Classical stability

Our lattice will be $K(\overrightarrow{Q}) \simeq \mathbb{Z}^I$. We first need to fix a linear functional $\theta : \mathbb{C}^I \to \mathbb{C}$.

Definition 3.12 (slope): Define the θ-**slope** of a $\overrightarrow{Q}$-representation V to be

$$\mu_\theta(V) = \frac{\theta(\mathbf{dim}\, V)}{\dim V},$$

where $\dim V := \sum_i (\mathbf{dim}\, V)_i$ is the total dimension of the vector spaces.

Definition 3.13 (classical (semi)stability): A representation V is **(classically)** μ**-semistable** if for every proper nonzero submodule $M \subset V$, then $\mu(M) \leq \mu(V)$. It is **stable** if additionally $\mu(M) < \mu(V)$.

Remark 3.14: This classical notion of (semi)stability is analogous to the classical notion of (semi)stable sheaves on smooth projective varieties.

3.6 The stability conditions agree

Fix $\theta = (\theta_i)_{i\in I}$ a linear functional on $\mathbb{C}^I$, and define μ_θ as above. Define the character of $\mathrm{GL}_{\mathbf{v}}$ by

$$\chi_\theta : \quad \mathrm{GL}_{\mathbf{v}} \ni (g_i)_{i\in I} \mapsto \prod_{i\in I} \mapsto \det(g_i)^{\mu_\theta(\mathbf{v})-\theta_i} \in \mathbb{C}^\times.$$

We have two notions of a representation V of graded dimension $\mathbf{v}$ being (semi)stable: one from the GIT sense, and one from the classical sense.

Theorem 3.15: A $\overrightarrow{Q}$-representation V of graded dimension $\mathbf{v}$ is χ_θ-(semi)stable in the GIT sense (as a point in $\mathcal{R}_\mathbf{v}$) iff it is μ_θ-(semi)stable in the classical sense.

Proof. We'll just prove it for semistable; the proof for stable is exactly the same, but replacing all of the $\leq$ with $<$. The key is to leverage Mumford's criterion (3.4) on the GIT side with filtrations on the classical side, so we need to understand how one-parameter subgroups interact with filtrations.

Lemma 3.16: Fix $V \in \mathcal{R}_\mathbf{v}$, a $\overrightarrow{Q}$-representation such that $\mathbf{dim}\, V = \mathbf{v}$. Let $V = \left(\{V_i\}_{i \in I}, \{\varphi_e\}_{e \in \Omega}\right)$.
To a one-parameter subgroup $\lambda : \mathbb{G}_m \to \mathrm{GL}_\mathbf{v}$ such that $\lim_{t \to 0} \lambda(t)$ exists, we obtain a finite filtration of V by subrepresentations. Conversely, to each (necessarily finite) filtration of V by subrepresentations, we obtain (non-uniquely) a one-parameter subgroup λ such that $\lim_{t \to 0} \lambda(t)$ exists.

Remark 3.17: We are not claiming that these are inverse operations; however, they are inverses in one direction: to a filtration of V, we produce a one-parameter subgroup λ whose limit exists, and the filtration we obtain from λ recovers our original filtration. The failure of the reverse composition is due to the choice of direct summand complement, so there are many one-parameter subgroups we could choose inducing the same filtration.

Proof. First suppose we have a one-parameter subgroup λ. We already have an action $\mathrm{GL}_\mathbf{v} \curvearrowright \mathcal{R}_\mathbf{v}$, hence λ induces an action of $\mathbb{G}_m$ on each V_i, $i \in I$. But a $\mathbb{G}_m$-action is the same as a $\mathbb{Z}$-grading, hence each V_i decomposes as $\bigoplus_{n \in \mathbb{Z}} V_i^{(n)}$, where $\lambda(t)|_{V_i^{(n)}} = t^n$. Write $V_i^{\geq n} := \bigoplus_{m \geq n} V_i^{(m)}$.
Now for each edge $e : i \to j \in \Omega$, the linear map $\varphi_e : V_i \to V_j$ decomposes into a direct sum $\varphi_e^{m,n} : V_i^{(n)} \to V_j^{(m)}$, with action of $\lambda(t)$ by

$$\lambda(t) \cdot \varphi_e^{m,n} = \lambda(t)|_{V_j} \cdot \varphi_e^{m,n} \cdot \lambda(t)|_{V_i}^{-1} = t^m \cdot \varphi_e^{m,n} \cdot t^{-n} = t^{m-n} \varphi_e^{m,n}.$$

So the limit $\lim_{t \to 0} \lambda(t)$ existing implies that for $m < n$, we have $\varphi_e^{m,n} = 0$, otherwise the λ-action blows $\varphi_e^{m,n}$ up to infinity. Thus φ_e always **increases** the weights (of the λ-action), hence we have well-defined maps $\varphi_e : V_i^{\geq n} \to V_j^{\geq n}$ for all $e \in \Omega$, and thus $V^{\geq n} := \left(\left\{V_i^{\geq n}\right\}_{i \in I}, \{\varphi_e\}\}_{e \in \Omega}\right)$ defines a subrepresentation. Thus from λ we obtain a filtration $\cdots \subseteq V^{\geq n+1} \subseteq V^{\geq n} \subseteq V^{\geq n-1} \subseteq \cdots$ of V by subrepresentations, and it must be finite because V is finite-dimensional.
On the other hand, let's suppose we have some finite filtration $V = V^k \supseteq V^{k+1} \supseteq \cdots \supseteq V^{k+n} = 0$ of V by subrepresentations. Then we can artificially construct a one-parameter subgroup (whose limit exists) by choosing some direct summand complement to each V^{i+1} in V^i, and declaring that $\lambda(t)$ acts on this direct summand complement by t^i. □

We also need to know one more thing: what $\langle \chi_\theta, \lambda \rangle$ is.

Lemma 3.18: Fix some V as before. Let λ be a one-parameter subgroup whose limit exists; by Lemma 3.6, we get an induced filtration by $V^{\geq n}$. Then $\langle \chi_\theta, \lambda \rangle = \sum_{n \in \mathbb{Z}} \left(\dim(V^{\geq n})\mu_\theta(\mathbf{v}) - \theta(\mathbf{dim}\, V^{\geq n})\right)$.

Proof. We can compute the composition $\chi_\theta \circ \lambda$ directly:

$$\chi_\theta(\lambda(t)) = \prod_{i\in I} \det(\lambda(t)_i)^{\mu_\theta(\mathbf{v})-\theta_i} = \prod_{i\in I}\prod_{n\in\mathbb{Z}} \det\left(\lambda(t)|_{V_i^{(n)}}\right)^{\mu_\theta(\mathbf{v})-\theta_i} = \prod_{i\in I}\prod_{n\in\mathbb{Z}} t^{n\cdot(\dim V_i^{(n)})\cdot(\mu_\theta(\mathbf{v})-\theta_i)}.$$

This computation tells us $\langle \chi_\theta, \lambda\rangle$:

$$\begin{aligned}\langle \chi_\theta, \lambda\rangle &= \sum_{i\in I}\sum_{n\in\mathbb{Z}} n\cdot(\dim V_i^{(n)})\cdot(\mu_\theta(\mathbf{v})-\theta_i),\\ &= \sum_{n\in\mathbb{Z}} n\cdot\left(\dim(V^{\geq n}/V^{\geq n+1})\chi_\theta(\mathbf{v}) - \theta(\mathbf{dim}\, V^{\geq n}/V^{\geq n+1})\right),\\ &= \sum_{n\in\mathbb{Z}} n\left(\dim(V^{\geq n})\mu_\theta(\mathbf{v}) - \theta(\mathbf{dim}\, V^{\geq n})\right) - n\left(\dim(V^{\geq n+1})\mu_\theta(\mathbf{v}) - \theta(\mathbf{dim}\, V^{\geq n+1})\right),\\ &= \sum_{n\in\mathbb{Z}} \left(\dim(V^{\geq n})\mu_\theta(\mathbf{v}) - \theta(\mathbf{dim}\, V^{\geq n})\right).\end{aligned}$$

□

Now let's return to the proof. Suppose V is χ_θ-semistable (in the GIT sense). We want to show that it is μ_θ-semistable (in the classical sense). So let $M \subset V$ be any proper nonzero subrepresentation, and treat this as the (very short) filtration $0 \subset M \subset V$. Then using Lemma 3.6, we can construct some one-parameter subgroup λ. Since V is χ_θ-semistable, Mumford's criterion (3.4) implies that $0 \leq \langle \chi_\theta, \lambda\rangle$. But in Lemma 3.6, we compute that

$$\begin{aligned} & 0 \leq \langle \chi_\theta, \lambda\rangle,\\ &\quad = \dim(0)\mu_\theta(\mathbf{v}) - \theta(0) + \dim(M)\mu_\theta(\mathbf{v}) - \theta(\mathbf{dim}\, M) + \dim(V)\mu_\theta(\mathbf{v}) - \theta(\mathbf{v}),\\ &\quad = \dim(M)\mu_\theta(\mathbf{v}) - \theta(\mathbf{dim}\, M),\\ \Longrightarrow\ & \mu_\theta(M) \leq \mu_\theta(\mathbf{v}) = \mu_\theta(V),\end{aligned}$$

so we conclude that V being χ_θ-semistable implies V is μ_θ-semistable.
Conversely, suppose V is μ_θ-semistable. To show that V is χ_θ-semistable, we just need to show that $\langle \chi_\theta, \lambda\rangle \geq 0$ for every λ whose limit exists. For any such λ, Lemma 3.6 gives us a filtration of V by subrepresentations $V^{\geq n}$. Then since V is μ_θ-semistable, we must have

$$\mu_\theta(V^{\geq n}) \leq \mu_\theta(V) = \mu_\theta(\mathbf{v})$$

for all n; this implies that

$$\dim(V^{\geq n})\mu_\theta(\mathbf{v}) - \theta(\mathbf{dim}\, V^{\geq n}) \geq 0.$$

Then Lemma 3.6 computes that

$$\langle \chi_\theta, \lambda\rangle = \sum_{n\in\mathbb{Z}} \left(\dim(V^{\geq n})\mu_\theta(\mathbf{v}) - \theta(\mathbf{dim}\, V^{\geq n})\right) \geq 0,$$

hence Mumford's criterion (3.4) implies that V is χ_θ-semistable. □

3.7 A worked-out example

Let's consider the quiver A_2:

$$A_2 := \bullet \longrightarrow \bullet.$$

There are two vertices, hence two simple representations, and so

$$K(A_2) \simeq \mathbb{Z}^2.$$

There are exactly **three** indecomposable representations, up to isomorphism:

- $V_1 \colon \mathbb{C} \to 0$.

- V_2: $0 \longrightarrow \mathbb{C}$.
- V_3: $\mathbb{C} \xrightarrow{\text{id}} \mathbb{C}$.

Note that V_1 and V_2 are the simple representations associated to the vertices, see (2.3).

Remark 3.19: In this case, A_2 is what's known as a *Dynkin quiver*, in that its underlying (undirected) graph is a Dynkin diagram. It corresponds to the simple Lie algebra $\mathfrak{sl}_3$, and it's known that the indecomposable representations are in bijection with the positive roots of $\mathfrak{sl}_3$, of which it has three. Furthermore, to a positive root $\alpha = \sum_{i \in I} n_i \alpha_i$, where α_i are the simple roots, the associated indecomposable representation has graded dimension $(n_i)_{i \in I}$. In our case, there are three positive roots: the simple roots α_1 and α_2 (which must have graded dimension $(1, 0)$ and $(0, 1)$, respectively), and the positive root $\alpha_1 + \alpha_2$ which has graded dimension $(1, 1)$.

Let us fix our graded dimension to be $\mathbf{v} = (1, 1)$, so that $\mathrm{GL}_{\mathbf{v}} = \mathrm{GL}_1 \times \mathrm{GL}_1 = \mathbb{C}^\times \times \mathbb{C}^\times$. We can easily see that there are exactly two representations of graded dimension $\mathbf{v}$: these are $V_1 \oplus V_2$ and V_3. Let us take $\theta = (a, b) \in \mathbb{Z}^2$ some arbitrary linear functional, and let's study when these representations are (semi)stable.

Example 3.20 (semistability for V_3): First, let's examine the **classical** case. First, we compute:

$$\mu_\theta(V_3) = \frac{\theta((1,1))}{1+1} = \frac{a+b}{2}.$$

Now the only subrepresentation of V_3 is V_2, so we need to check that $\mu_\theta(V_2) \le \mu_\theta(V_3)$. We have

$$\mu_\theta(V_2) = \frac{\theta((0,1))}{0+1} = b,$$

so

$$V_3 \text{ is } \mu_\theta\text{-semistable} \iff \mu_\theta(V_2) \le \mu_\theta(V_3) \iff b \le a.$$

Now let's look at the **GIT** side. Our character is

$$\chi_\theta : \quad \mathbb{C}^\times \times \mathbb{C}^\times \ni (s, t) \mapsto g^{\frac{a+b}{2} - a} \cdot s^{\frac{a+b}{2} - b} = \left(\frac{s}{g}\right)^{\frac{a-b}{2}}.$$

Now a one-parameter subgroup $\lambda : \mathbb{C}^\times \to \mathbb{C}^\times \times \mathbb{C}^\times$ is just a product of two characters, $t \mapsto (t^m, t^n)$; so take $\lambda = (m, n)$. So we started with $V_3 = \left(\mathbb{C} \xrightarrow{\cdot 1} \mathbb{C}\right)$; we compute that

$$\lambda(t) \cdot V_3 = \left(\mathbb{C} \xrightarrow{\cdot t^{n-m}} \mathbb{C}\right).$$

It follows that the limit $\lim_{t \to 0} \lambda(t) \cdot V_3$ exists iff $n \ge m$; so we need only consider one-parameter subgroups λ corresponding to (n, m) with $n \ge m$. Now we just compute that

$$\chi_\theta \circ \lambda : \quad t \mapsto \left(\frac{t^n}{t^m}\right)^{\frac{a-b}{2}} = t^{(n-m)(a-b)/2}.$$

Then

$$V_3 \text{ is } \chi_\theta\text{-semistable} \iff \langle \lambda_{m,n}, \chi_\theta \rangle \ge 0 \text{ for all } n \ge m \iff \frac{(n-m)(a-b)}{2} \ge 0 \text{ for all } n \ge m \iff a \ge b.$$

So we conclude that the two notions of stability are indeed exactly the same here.

Example 3.21 (stability for V_3): Running through the previous argument, we have

$$V_3 \text{ is } \mu_\theta\text{-stable} \iff a > b,$$

and

$$V_3 \text{ is } \chi_\theta\text{-stable} \iff \frac{(n-m)(a-b)}{2} > 0 \text{ for all } n > m \iff a > b.$$

So once again, they agree. (Note that this time, we need to use the modified version of Mumford's criterion (3.4), which requires λ to be *nontrivial*, which is equivalent to $n > m$.)

Example 3.22 (semistability for $V_1 \oplus V_2$): Let's again start with the classical case. We have two subrepresentations of $V_1 \oplus V_2$, namely V_1 and V_2. Then $\mu_\theta(V_1) = a$, $\mu_\theta(V_2) = b$, and $\mu_\theta(V_1 \oplus V_2) = \frac{a+b}{2}$. Therefore

$$V_1 \oplus V_2 \text{ is } \mu_\theta\text{-semistable} \iff \mu_\theta(V_1), \mu_\theta(V_2) \le \mu_\theta(V_1 \oplus V_2) \iff a = b.$$

Now let's look at the GIT side. Note that we started with

$$V_1 \oplus V_2 = \left(\mathbb{C} \xrightarrow{0} \mathbb{C}\right),$$

so for *any* one-parameter subgroup λ, then $\lambda(t)$ does nothing to $V_1 \oplus V_2$. Therefore the limit always exists, and so

$$V_1 \oplus V_2 \text{ is } \chi_\theta\text{-semistable} \iff 0 \le \langle \lambda, \chi_\theta \rangle \text{ for all } \lambda \iff \frac{(n-m)(a-b)}{2} \ge 0 \text{ for all } n, m \iff a = b.$$

So once again, the two notions agree.

Example 3.23 (stability for $V_1 \oplus V_2$): To be μ_θ semistable, we'd need both $a > b$ *and* $b > a$, which is impossible, so actually $V_1 \oplus V_2$ is **never** μ_θ-semistable (for any θ).
On the other hand, since every one-parameter subgroup acts trivially, $V_1 \oplus V_2$ cannot be χ_θ-semistable.

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References

[Bal] Ana Balibanu. Semistable representations of quivers.

[KJ16] Alexander Kirillov Jr. *Quiver representations and quiver varieties*, volume 174. American Mathematical Soc., 2016.